Introduction to the L2E Package (version 2.0)

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Overview

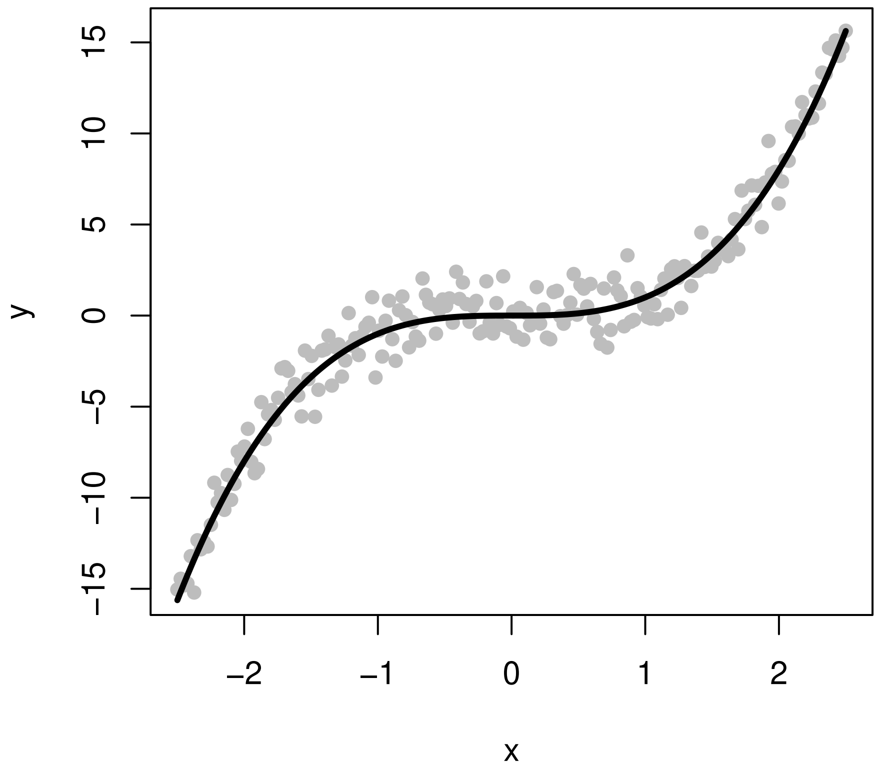
The `L2E` package (version 2.0) implements the computational framework for L_2E regression in Liu, Chi, and Lange (2022), which was built on the previous work in Chi and Chi (2022). Both work use the block coordinate descent strategy to solve a nonconvex optimization problem but employ different methods for the inner block descent updates. We refer to the method in Liu, Chi, and Lange (2022) as "MM" and the one in Chi and Chi (2022) as "PG" in our package. This vignette provides code to replicate some examples illustrating the usage of the frameworks in both papers and the improvements from Liu, Chi, and Lange (2022).

Examples

Isotonic Regression

The first example illustrates how to perform robust isotonic regression via the L_2 criterion, also called L_2E isotonic regression. We begin by generating the true fit f from a cubic function (depicted by the black line on the figure below). We obtain the observed response by adding some Gaussian noise to f (depicted by the gray points on the figure below).

```r
set.seed(12345)
n <- 200
tau <- 1
x <- seq(-2.5, 2.5, length.out=n)
f <- x^3
y <- f + (1/tau)*rnorm(n)
plot(x, y, pch=16, col='gray')
lines(x, f, lwd=3)
```



The `L2E_isotonic` function provides two options for implementing the L_2E isotonic regression. By setting the argument "method = MM" (the default), the function calls `l2e_regression_isotonic_MM` to perform L_2E isotonic regression using the MM method in Liu, Chi, and Lange (2022). Setting "method = PG" leads the function to call `l2e_regression_isotonic` to perform L_2E isotonic regression using the PG method in Chi and Chi (2022).

We obtain and compare three estimates for the underlying fit f from the classical least squares (LS), the L_2E with MM, and the L_2E with PG, respectively. The true fit is shown in black while the fits from the LS, the L_2E with MM, and the L_2E with PG are shown in blue, red, and green, respectively. In the absence of outliers, all methods produce similar estimates. For the two L_2E methods, MM is faster than PG.

```
library(L2E)
```

```
Loading required package: osqp
```

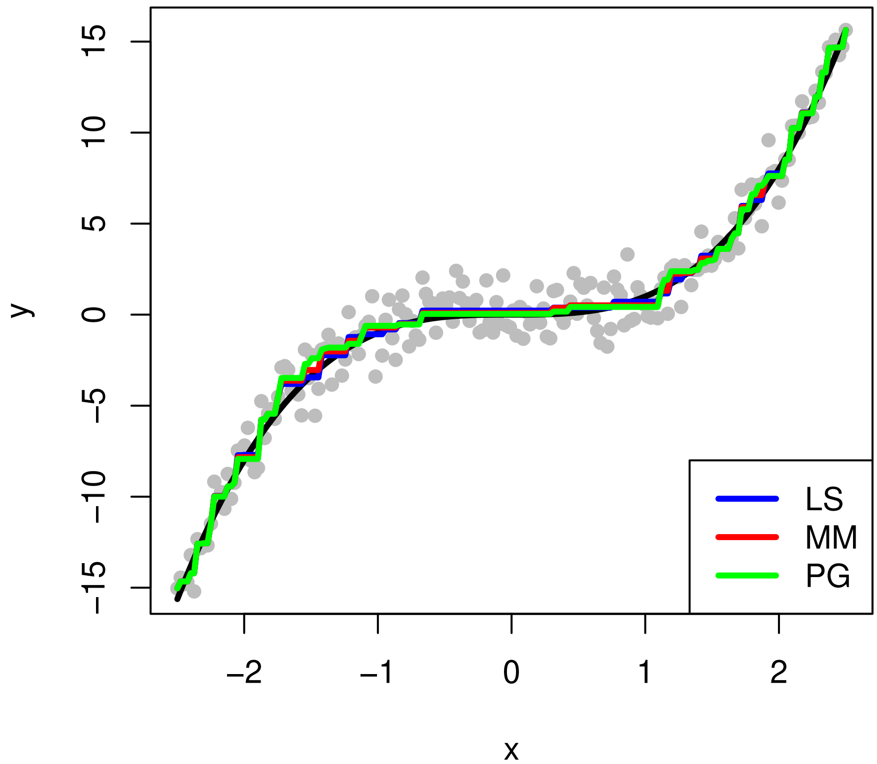
```
library(isotone)
tau <- 1/mad(y)
b <- y
# LS method
iso <- gpava(1:n, y)$x
# MM method
sol_mm <- L2E_isotonic(y, b, tau, method = "MM")
```

```
   user  system elapsed
  0.169   0.000   0.169
```

```
# PG method
sol_pg <- L2E_isotonic(y, b, tau, method = 'PG')
```

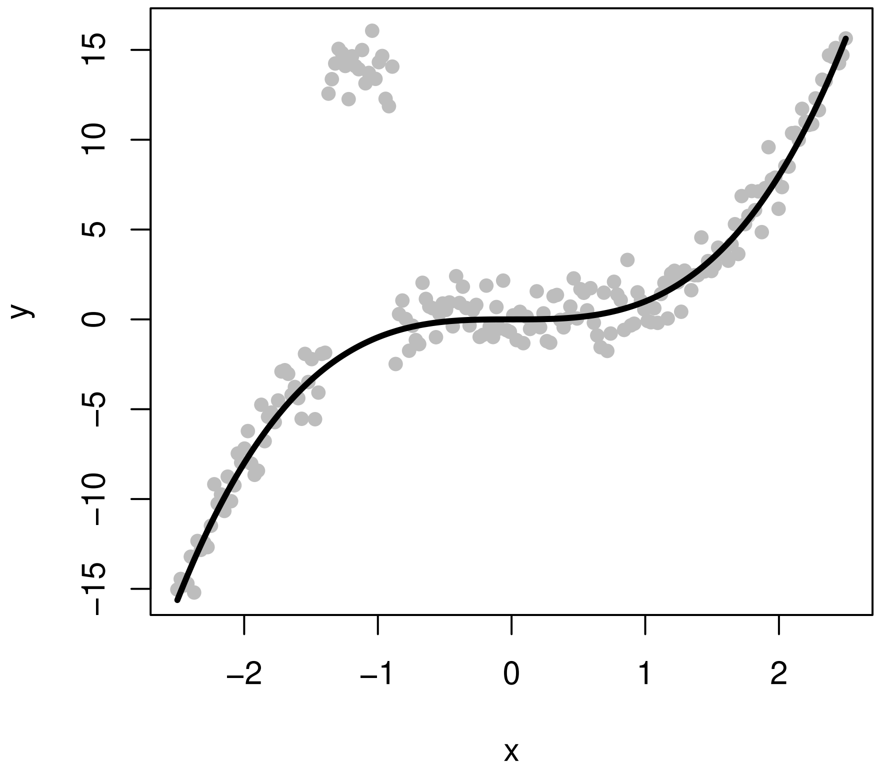
```
   user  system elapsed
  0.303   0.007   0.311
```

```r
# Plots
plot(x, y, pch=16, col='gray')
lines(x, f, lwd=3)
lines(x, iso, col='blue', lwd=3) ## LS
lines(x, sol_mm$beta, col='red', lwd=3) ## MM
lines(x, sol_pg$beta, col='green', lwd=3) ## PG
legend("bottomright", legend = c("LS", "MM", "PG"), col = c('blue','red', 'green'), lwd=3)
```



Next, we introduce some outliers by perturbing some of the observed responses.

```r
num <- 20
ix <- 1:num
y[45 + ix] <- 14 + rnorm(num)
plot(x, y, pch=16, col='gray')
lines(x, f, lwd=3)
```



We obtain the three estimates with the outlying points. The L_2E fits are less sensitive to outliers than the LS fit in this example. The two methods MM and PG produce comparable estimates, but MM is faster than PG.

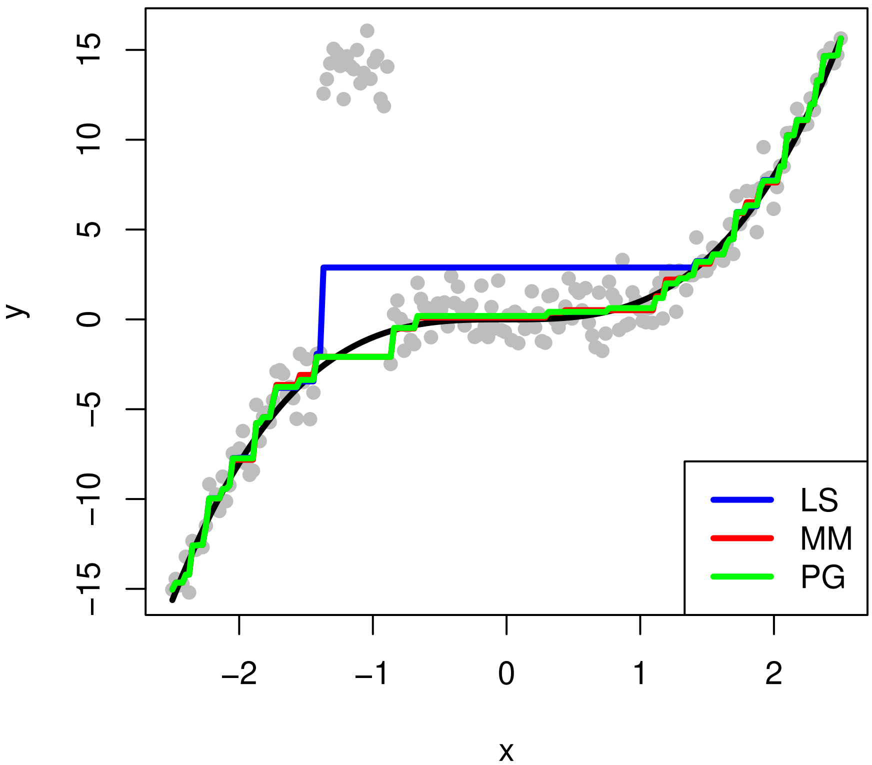
```
tau <- 1/mad(y)
b <- y
# LS method
iso <- gpava(1:n, y)$x
# MM method
sol_mm <- L2E_isotonic(y, b, tau, method = "MM")
```

```
   user  system elapsed 
  0.036   0.000   0.036 
```

```
# PG method
sol_pg <- L2E_isotonic(y, b, tau, method = 'PG')
```

```
   user  system elapsed 
  0.178   0.000   0.177 
```

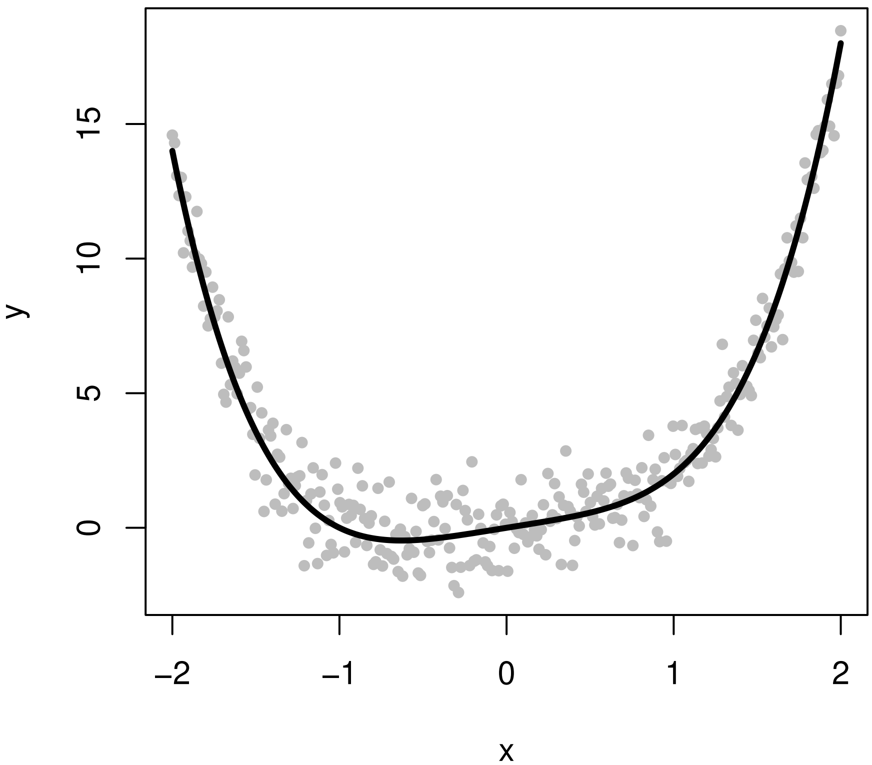
```
# Plots
plot(x, y, pch=16,  col='gray')
lines(x, f, lwd=3)
lines(x, iso, col='blue', lwd=3) ## LS
lines(x, sol_mm$beta, col='red', lwd=3) ## MM
lines(x, sol_pg$beta, col='green', lwd=3) ## PG
legend("bottomright", legend = c("LS", "MM", "PG"), col = c('blue','red', 'green'), lwd=3)
```



Convex Regression

The second example illustrates how to perform robust convex regression via the L_2 criterion, also called L_2E convex regression. We begin by simulating a convex function f. The observed response is composed of f with some additive Gaussian noise.

```
set.seed(12345)
n <- 300
tau <- 1
x <- seq(-2, 2, length.out=n)
f <- x^4 + x
y <- f + (1/tau) * rnorm(n)
plot(x, y, pch=16, col='gray', cex=0.8)
lines(x, f, col='black', lwd=3)
```



The L2E_convex function provides two options for implementing the L_2E convex regression. By setting the argument "method = MM" (the default), the function calls l2e_regression_convex_MM to perform L_2E convex regression using the MM method in Liu, Chi, and Lange (2022). Setting "method = PG" leads the function to call l2e_regression_convex to perform L_2E convex regression using the PG method in Chi and Chi (2022).

We obtain and compare three estimates for the underlying fit f from the classical least squares (LS), the L_2E with MM, and the L_2E with PG, respectively. The true fit is shown in black while the fits from the LS, the L_2E with MM, and the L_2E with PG are shown in blue, red, and green, respectively. In the absence of outliers, all methods produce similar estimates. For the two L_2E methods, MM is faster than PG.

```
library(cobs)
tau <- 1/mad(y)
b <- y
## LS method
cvx <- fitted(conreg(y, convex=TRUE))
## MM method
sol_mm <- L2E_convex(y, b, tau, method = "MM")
```

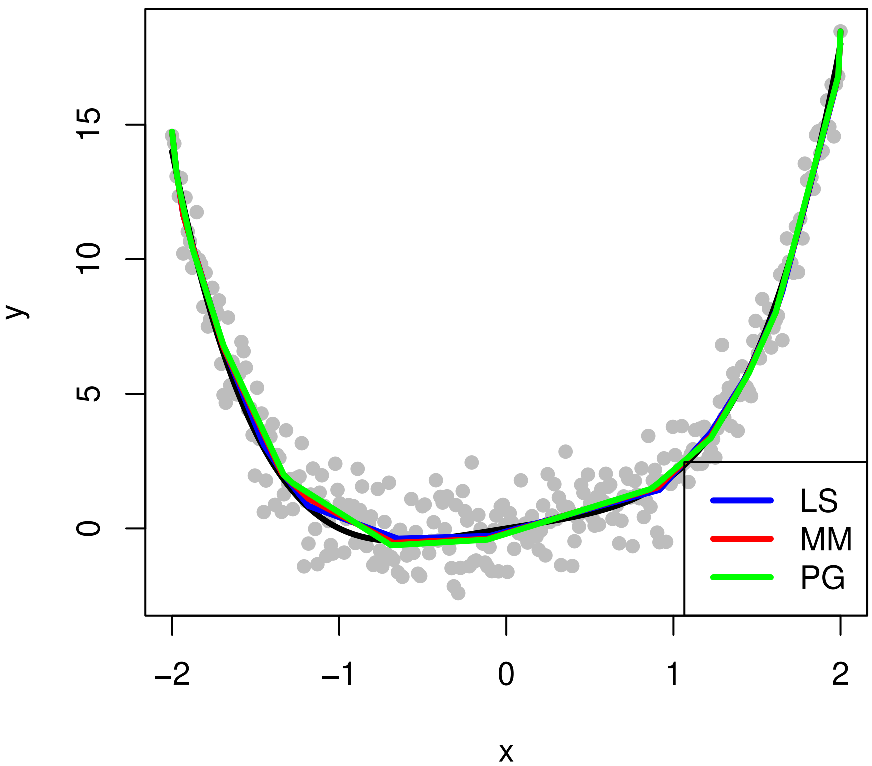
```
   user  system elapsed 
  0.133   0.008   0.142 
```

```
## PG method
sol_pg <- L2E_convex(y, b, tau, method = 'PG')
```

```
   user  system elapsed 
  1.060   0.004   1.064 
```

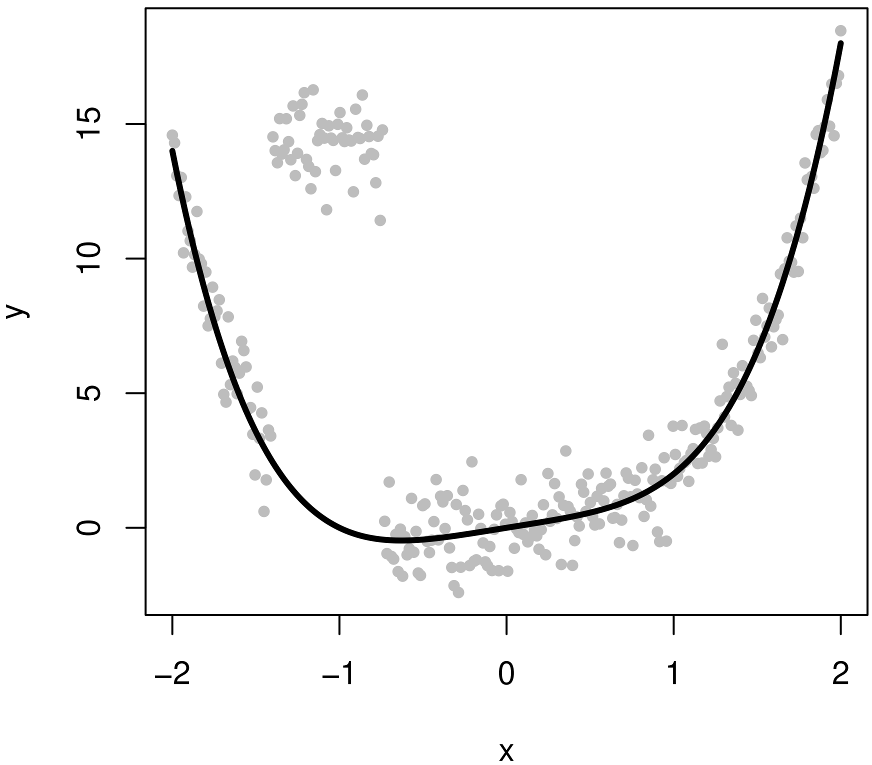
```
plot(x, y, pch=16, col='gray')
lines(x, f, lwd=3)
lines(x, cvx, col='blue', lwd=3) ## LS
lines(x, sol_mm$beta, col='red', lwd=3) ## MM
```

```
lines(x, sol_pg$beta, col='green', lwd=3) ## PG
legend("bottomright", legend = c("LS", "MM", "PG"), col = c('blue','red', 'green'), lwd=3)
```



Next, we again introduce some outlying points.

```
num <- 50
ix <- 1:num
y[45 + ix] <- 14 + rnorm(num)

plot(x, y, pch=16, col='gray', cex=0.8)
lines(x, f, col='black', lwd=3)
```



We obtain the three estimates with the outlying points. The L_2E fits are less sensitive to outliers than the LS fit in this example. The MM method not only requires less computional time but also produces better estimation than the PG method.

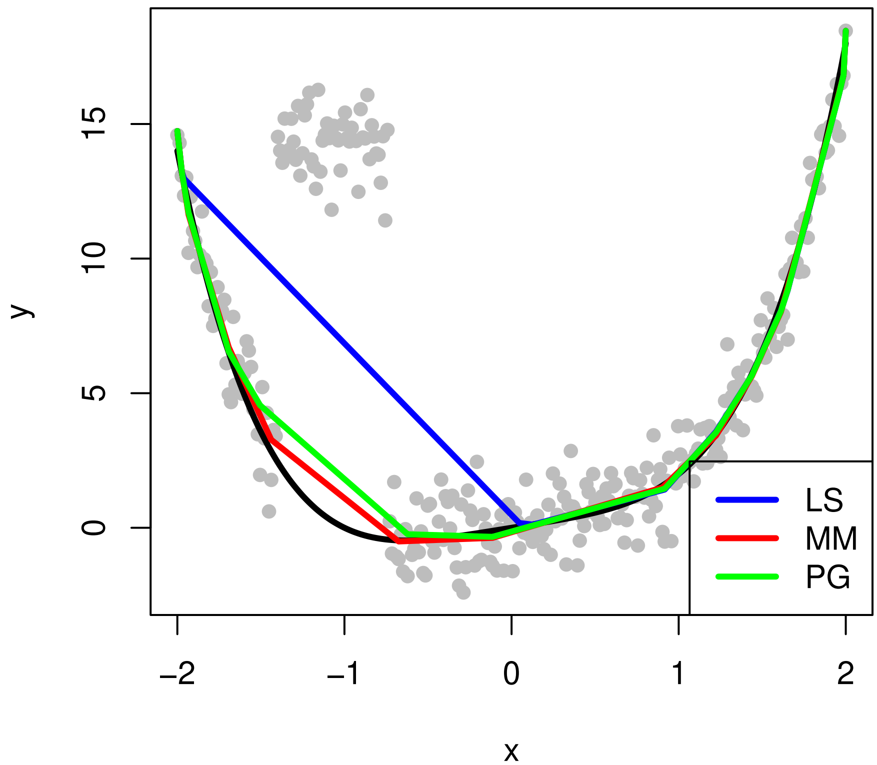
```
tau <- 1/mad(y)
b <- y
## LS method
cvx <- fitted(conreg(y, convex=TRUE))
## MM method
sol_mm <- L2E_convex(y, b, tau, method = "MM")
```

```
   user  system elapsed 
  0.167   0.000   0.167 
```

```
## PG method
sol_pg <- L2E_convex(y, b, tau, method = 'PG')
```

```
   user  system elapsed 
  2.738   0.000   2.739 
```

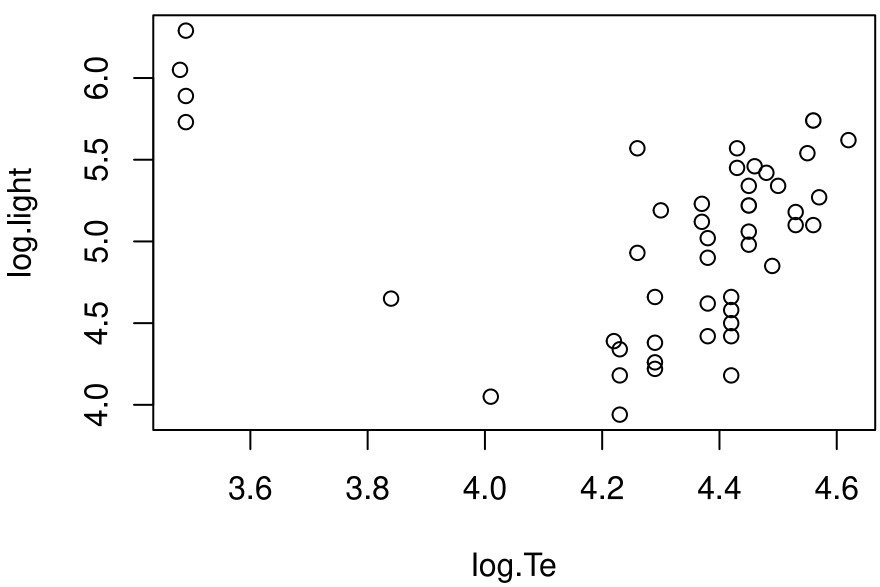
```
plot(x, y, pch=16, col='gray')
lines(x, f, lwd=3)
lines(x, cvx, col='blue', lwd=3) ## LS
lines(x, sol_mm$beta, col='red', lwd=3) ## MM
lines(x, sol_pg$beta, col='green', lwd=3) ## PG
legend("bottomright", legend = c("LS", "MM", "PG"), col = c('blue','red', 'green'), lwd=3)
```



Multivariate Regression

The third example provides code to perform L_2E multivariate regression with the Hertzsprung-Russell diagram data of star cluster CYG OB1. The data set is commonly used in robust regression owing to its four known outliers – four bright giant stars observed at low temperatures. The data set is publicly available in the R package `robustbase`. We begin by loading the data from the `robustbase` package.

```
library(robustbase)
data(starsCYG)
plot(starsCYG)
```



We use the `L2E_multivariate` function to perform the L_2E multivariate reeegression, namely, multivariate regression via the L_2 criterion. The `L2E_multivariate` function provides two options for implementing the L_2E multivariate regression. By setting the argument "method = MM" (the default), the function calls `l2e_regression_MM` to perform L_2E multivariate regression using the MM method in Liu, Chi, and Lange (2022). Setting "method = PG" leads the function to call `l2e_regression` to perform L_2E multivariate regression using the PG method in Chi and Chi (2022).

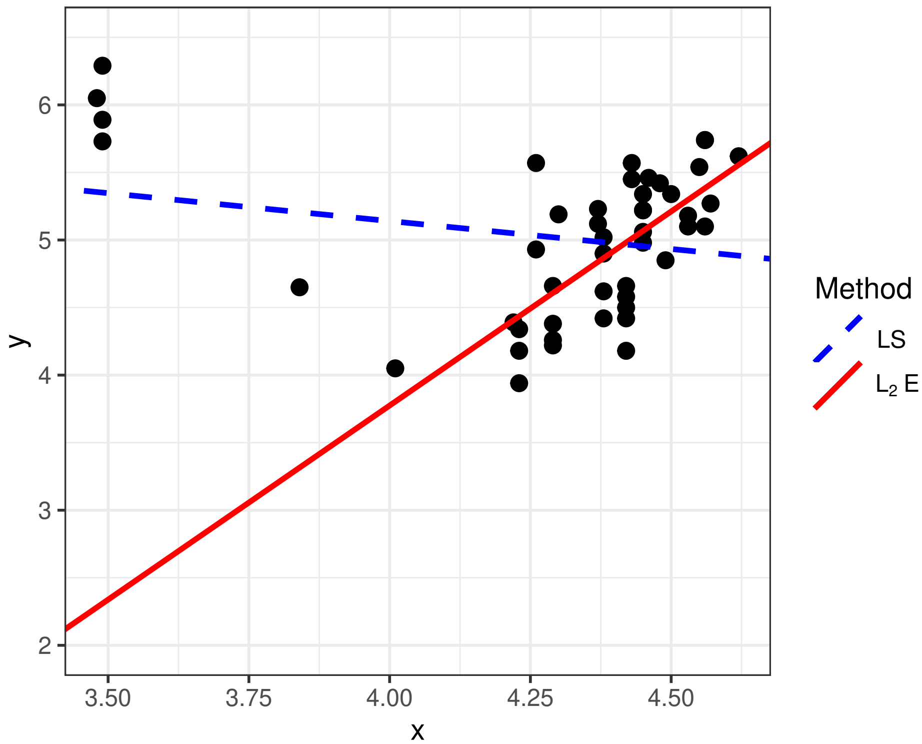
We next use the MM method to fit the L_2E multivariate regression model and compare it to the least squares fit. The figure below shows that the L_2E fit can reduce the influence of the four outliers and fits the remaining data points well. The LS fit is heavily affected by the four outliers.

```
y <- starsCYG[, "log.light"]
x <- starsCYG[, "log.Te"]
X0 <- cbind(rep(1, length(y)), x)

# LS method
mle <- lm(log.light ~ log.Te, data = starsCYG)
r_lm <- y - X0 %*% mle$coefficients

# L2E+MM method
tau <- 1/mad(y)
b <- c(0, 0)
# Fit the regression model
sol_mm <- L2E_multivariate(y, X0, b, tau, method="MM")
```

```
   user  system elapsed 
  0.014   0.000   0.014 
```

```
l2e_fit_mm <- X0 %*% sol_mm$beta
# compute limit weights
r_mm <- y - l2e_fit_mm


data <- data.frame(x, y, l2e_fit_mm)


d_lines <- data.frame(int = c(sol_mm$beta[1], mle$coefficients[1]),
                      sl = c(sol_mm$beta[2], mle$coefficients[2]),
                      col = c("red", "blue"),
                      lty = c("solid", "dashed"),
                      method = c("L2E", "LS"))

ltys <- as.character(d_lines$lty)
names(ltys) <- as.character(d_lines$lty)

cols <- as.character(d_lines$col)
cols <- cols[order(as.character(d_lines$lty))]
method <-  as.character(d_lines$method)


library(ggplot2)
library(latex2exp)
p <- ggplot() +
  geom_point(data = data, aes(x, y), size=2.5) + ylim(2, 6.5)+
  geom_abline(data = d_lines[d_lines$col == "red", ],
              aes(intercept = int, slope = sl, lty = lty), color = "red", size=1) +
  geom_abline(data = d_lines[d_lines$col == "blue", ],
```

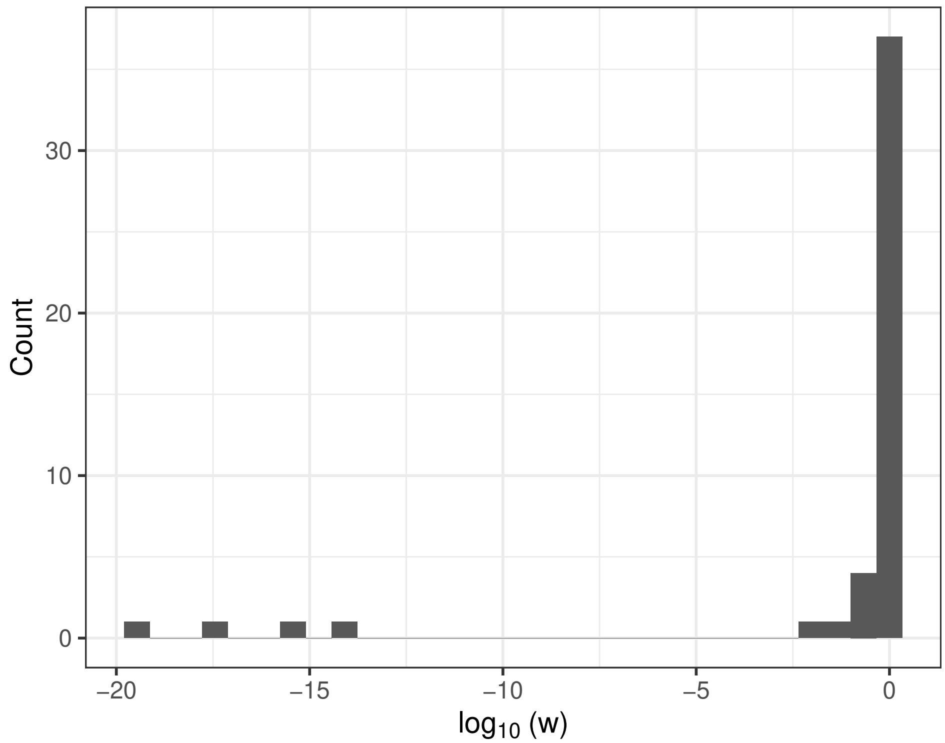
```
               aes(intercept = int, slope = sl, lty = lty), color = "blue", size=1) +
  scale_linetype_manual(name = "Method", values = ltys, breaks=c("dashed", "solid"),
                        labels = c("LS  ",
                                   expression(L[2]~E)),
                        guide = guide_legend(override.aes = list(colour = cols), legend=method))+
  theme_bw()

print(p)
```



We next show how to use the convereged weights from the MM method to identify outliers in the data. As discussed in Liu, Chi, and Lange (2022), a small weight suggests a potential outlier.

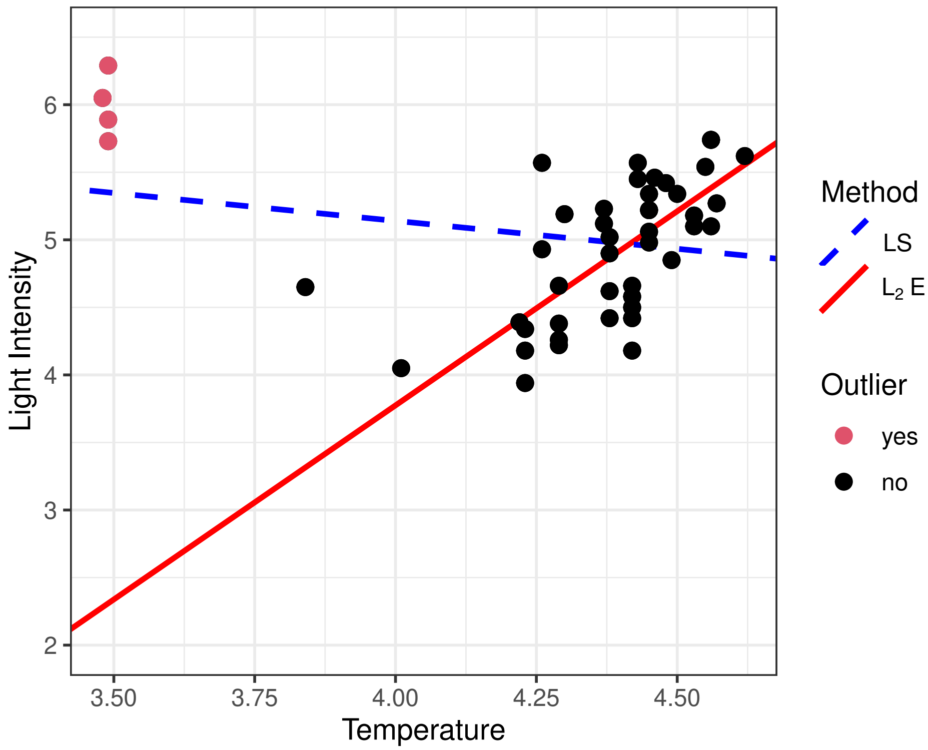
```
w <- as.vector(exp(-0.5* (sol_mm$tau*r_mm)**2 ))
data <- data.frame(x, y, l2e_fit_mm, w)
ggplot(data, aes(x=log10(w))) + geom_histogram()+
  labs(
    y="Count", x=expression(log[10]~'(w)'))+theme_bw()
```

```
`stat_bin()` using `bins = 30`. Pick better value with `binwidth`.
```



The four extremely small weights in the above histogram indicate that there are four potential outliers in the data. Next, we incorporate the four outliers into the scatter plot. The MM method successfully identify the four outliers.

```
outlier_mm <- rep("yes", length(y))
for (k in 1:length(y)) {
  if(w[k]>1e-5) # the threshold value can range from 1e-3 to 1e-14 according to the histogram
    outlier_mm[k] <- "no"
}
outlier_mm  <- factor(outlier_mm , levels=c("yes", "no"))
data <- data.frame(x, y, l2e_fit_mm, outlier_mm)


p+
  geom_point(data = data, aes(x, y, color=outlier_mm), size=2.5) +
  scale_color_manual(values = c(2,1), name="Outlier")+
  labs(
    y="Light Intensity", x="Temperature")+ theme_bw()
```



Other Examples

The new `L2E` package include implementation of L_2E sparse regression and L_2E trend filtering. The `L2E_sparse_ncv` function computes a solution path of L_2E sparse regression with exisitng penalties available in the `ncvreg` package. The `L2E_sparse_dist` function computes a solution path of L_2E sparse regression with the distance penalty. The `L2E_TF_lasso` function computes a solution path of L_2E trend filtering with the Lasso penalty. The `L2E_TF_dist` function computes a solution path of L_2E trend filtering with the distance penalty. Readers may refer to Liu, Chi, and Lange (2022) for advantages of distance penalization in the two examples of sparse regression and trend filtering.

References

Chi, Jocelyn T., and Eric C. Chi. 2022. "A User-Friendly Computational Framework for Robust Structured Regression with the l_2 Criterion." *Journal of Computational and Graphical Statistics*, 1–12.

Liu, Xiaoqian, Eric C. Chi, and Kenneth Lange. 2022. "A Sharper Computational Tool for l_2e Regression." *arXiv Preprint arXiv:2203.02993*.